The Bootstrap: Theory and Applications

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Notes:

Motivation: Poor Asymptotic Approximation

- Most of statistical inference relies on asymptotic theory.
  eg. t-ratio $\xrightarrow{d}$ N(0,1),

  Wald test, Lagrange Multiplier test, and likelihood ratio test $\xrightarrow{d} \chi^2(d.f.)$ as sample goes to infinity.

Notes:

Motivation: Poor Asymptotic Approximation

- The idea is at its best hope the finite-sample distribution of the statistics COULD be well approximated by their asymptotic counterparts.
- But the reality in some cases might not turn out to be hoped.

Notes:

Motivation: Poor Asymptotic Approximation

- Simulated illustration

$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + e_i$$
$$\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} \sim N(0, I_2)$$
$$e_i \sim N(0, 9)$$
$$\beta_0 = 0, \beta_1 = 1, \beta_2 = 0.5, n(\text{sample size}) = 300$$

Notes:

Motivation: Poor Asymptotic Approximation

- The parameter of interest

$$\theta = \frac{\beta_1}{\beta_2}$$

which is estimated by

$$\widehat{\theta} = \frac{\widehat{\beta}_1}{\widehat{\beta}_2},$$

where $\widehat{\beta}_1, \widehat{\beta}_2$ are OLS estimates.

Notes:

- Want to test

$$H_0: \quad \theta_0 = 2 \text{ (true value)}$$

by using t-ratio

$$t_{300} = \frac{(\widehat{\theta} - \theta_0)}{S(\widehat{\theta})}$$

where

$$S(\widehat{\theta}) = (\widehat{H}'_\beta \widehat{V} \widehat{H}_\beta)^{1/2}, \widehat{H}'_\beta = (0, \frac{1}{\widehat{\beta}_2}, -\frac{\widehat{\beta}_1}{\widehat{\beta}_2}),$$

$\widehat{V}$: covariance matrix

based on "Taylor expansion" or "delta method".

Notes:

Motivation: Poor Asymptotic Approximation

- Exact distribution of t_{300} can be calculated by simulation.
  Replications: 100,000.
- We found dramatic divergence between the exact and asymptotic distribution.
  Exact: skewed and non-normal
  Asym.: symmetric and normal

Notes:

Motivation: Poor Asymptotic Approximation

- Pr(t_{300} >1.645)=0.00, Pr(t_{300} <-1.645)=0.115, and Pr($|t_{300}|$ >1.96)=0.084

  indicative of a significant "type I error" distortion.

Notes:

The Bootstrap as an Alternative Approximation to Dist. of Statistics

- It is a re-sampling scheme by treating the data as if they were the population.
- It is to estimate the finite-sample distribution of a statistic, and thus moments or quantiles.
- It is often more accurate than first-order approximations, the so-called asymptotic refinement.

Notes:

Notations and Definition of the Problem

- The data: $\{x_i\}_{i=1}^{n} \sim F_0$, an unknown CDF.
- The statistic $T_n = T_n(x_1, x_2, \ldots, x_n)$ whose finite-sample CDF is denoted by $G_n(\tau, F_0) = Pr(T_n \leq \tau)$.
- $G_n(\tau, F)$ is a different function of τ for different F.

Notes:

Notations and Definition of the Problem

- T_n is "pivotal" when $G_n(\tau, F)$ does not depend on F.
- In most applications, pivotal statistics are not available.

Notes:

Notations and Definition of the Problem

- Therefore, $G_n(., F_0)$ usually depends on F_0, and also CAN'T be calculated, due to unknown F_0.
- Both the bootstrap and the asymptotic theory are ways to estimate $G_n(., F_0)$.

Notes:

Important Notion

- The limit distributions of many econometric statistics are $N(0,1)$ or χ^2, independent of F_0 or unknown population parameters.
- These statistics are called "asymptotically pivotal".
- In notation, let $G_\infty(., F_0)$ denote the asymptotic CDF of T_n.
- If T_n is asymptotically pivotal,

$$G_\infty(., F_0) \equiv G_\infty(.)$$

Notes:

Important Notion

- The idea of asymptotic approximation is when n is large enough, $G_\infty(.)$ is used to estimate $G_n(., F_0)$ without knowing F_0.
- As illustrated before, $G_\infty(.)$ can be a very poor approximation.

Notes:

The Idea of the Bootstrap

- The bootstrap replaces F_0 with an estimator F_n.
  (Note: asymptotic approximation replaces G_n with G_∞.)
- A nature choice of F_n:
  Empirical distribution function of the data

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x)$$

with I being the indicator function.

Notes:

The Idea of the Bootstrap

- Theoretical underpinning of the choice

$$\sup_x |F_n(x) - F_0(x)| \xrightarrow{p} 0$$

(by Glivenko-Cantelli theorem)

- So the bootstrap estimator of $G_n(., F_0)$ is $G_n(., F_n)$.

Notes:

Monte-Carlo Procedures to Produce $G_n(., F_n)$

- Step 1. Generate a bootstrap sample $\{x_i^*\}_{i=1}^n$ from the EDF $\{x_i\}_{i=1}^n$ with replacement.
- Step 2. Compute $T_n^* = T_n(x_1^*, \ldots, x_n^*)$.
- Step 3. Repeat Step 1, 2 B times, and then compute $Pr(T_n^* \leq \tau)$.

Notes:

Consistency of the Bootstrap

- The bootstrap estimator $G_n(., F_n)$ is consistent if for each $\epsilon > 0$,

$$\lim_{n\to\infty} Pr\left[\sup_{\tau} |G_n(\tau, F_n) - G_\infty(\tau, F_0)| > \epsilon\right] = 0.$$

- This is the minimal criterion for $G_n(., F_n)$ to be an adequate estimator.
- The bootstrap consistency is expected in most econometric applications.

Notes:

Subsampling

- This is an alternative re-sampling method.
- It often estimates $G_n(., F_0)$ consistently even when the bootstrap does not.
- It is not a perfect substitute for the bootstrap, because it tends to be less accurate than the bootstrap.

Notes:

Subsampling: Intuition and Operation

- $t_n = t_n(x_1, \ldots, x_n)$: an estimator of the parameter θ.
- Let $T_n = \rho(n)(t_n - \theta)$, where $\rho(n)$ is the normalization factor.
  eg. θ is the population mean, $t_n = \bar{x}$, $\rho(n) = n^{1/2}$.

Notes:

Subsampling: Intuition and Operation

- Let $\{x_{i_j}\}_{j=1}^{m}$ be a subset of $m < n$ drawn from $\{x_i\}_{i=1}^{n}$.
- $N_{n,m} = C_m^n$: total sub-sample numbers.
- $t_{m,k}$: the estimator t_m evaluated at the k^{th} subsample.

Notes:

Subsampling: Intuition and Operation

- The subsampling method estimates $G_n(\tau, F_0)$ by

$$G_{n,m}(\tau) = \frac{1}{N_{n,m}} \sum_{k=1}^{N_{n,m}} I[\rho(m)(t_{m,k} - t_n) \leq \tau]$$

- Intuition:
  $G_m(., F_0)$ is the exact sampling distribution of $\rho(m)(t_m - \theta)$, and

$$\begin{aligned} G_m(\tau, F_0) &= Pr(\rho(m)(t_m - \theta) \leq \tau) \\ &= E\{I[\rho(m)(t_m - \theta) \leq \tau]\}. \end{aligned}$$

Notes:

Subsampling: Intuition and Operation

- $G_{n,m}(\tau)$ would estimate $G_m(\tau, F_0)$ well, because if n is large but m/n is small (so $N_{n,m}$ is large), t_n varies less than t_m, and $\rho(t_m - t_n)$ and $\rho(m)(t_m - \theta)$ are close.

- Expectedly,
  $G_{n,m}(\tau) \sim G_m(\tau, F_0) \sim G_\infty(\tau, F_0)$
  as $n \to \infty$, $m \to \infty$.

- Applications: Confidence interval for highly persistent AR coefficient.

Notes:

Asymptotic Refinement

- Under some regularity conditions,

$$G_n(\tau, F_n) - G_n(\tau, F_0) = [G_\infty(\tau, F_n) - G_\infty(\tau, F_0)] + O(\frac{1}{n}).$$

- If T_n is not asymptotically pivotal,

$$G_\infty(\tau, F_n) - G_\infty(\tau, F_0) = O(n^{-1/2})$$

  because $F_n - F_0 = O(n^{-1/2})$.
  The bootstrap has the same accuracy as asymptotic approximations.

Notes:

Asymptotic Refinement

- If T_n is asymptotically pivotal,

$$G_n(\tau, F_n) - G_n(\tau, F_0) = G_\infty(\tau) - G_\infty(\tau)] = 0.$$

  The bootstrap now makes an error of size $O(n^{-1})$, smaller than that by asymptotic approximations $O(n^{-1/2})$.

- Note: The argument is valid for stationary data, but not necessarily true for nonstationary data.

- The smoothness condition has to be satisfied for the statistics to be asymptotically pivotal.

Notes:

Application: Bias Reduction

- Step 1: Use $\{x_i\}_{i=1}^n$ to yield θ_n, the estimated parameter.
- Step 2: Generate a bootstrap sample $\{x_i^*\}_{i=1}^n$ from $\{x_i\}_{i=1}^n$ with replacement. Use $\{x_i^*\}_{i=1}^n$ to compute θ_n^*.
- Step 3: Compute $E^*\theta_n^*$ by averaging B repetitions of θ_n^* in Step 2. Set $B_n^* = E^*\theta_n^* - \theta_n$.
- Step 4: Form the bias-corrected estimate by computing $\theta_n - B_n^*$.

Notes:

Application: Computing Bootstrap Critical Values

- Step 1-3 are the same as before, and set $Z_{n,\alpha}^*$ equal to the $(1-\alpha)$ quantile of the bootstrap distribution of T_n^*.

Notes:

Application: Constructing Confidence Interval

- Given $n^{1/2}(\theta_n - \theta_0)/S_n \xrightarrow{d} N(0,1)$, Where S_n is a consistent estimate of std.
- An asymptotic $(1-\alpha)$ 2-sided confidence interval for θ_0 is

$$\left[\theta_n - Z_{\infty,\alpha/2} S_n n^{-1/2}, \theta_n + Z_{\infty,\alpha/2} S_n n^{-1/2}\right].$$

- The bootstrap version is

$$\left[\theta_n - Z^*_{n,\alpha/2} S_n n^{-1/2}, \theta_n + Z^*_{n,\alpha/2} S_n n^{-1/2}\right].$$

Notes:

Bootstrap Sampling with Dependent Data

- Asymptotic refinement can't be obtained by independent bootstrap sampling, with dependent data.
- Bootstrap sampling has to be conducted in a way that suitably captures the dependence of the DGP.

Notes:

Dependent Data: Block Bootstrap

- Step 1: Pick a block length $b < n$.
- Step 2: Form n blocks of b residuals, each starting with a different one of the n residuals, and wrapping around to the beginning if necessary.
- Step 3: Generate the bootstrap errors by resampling the blocks. If n/b is an integer, there will have n/b blocks. Otherwise, the last block will have to be shorter than b.

Notes:

Dependent Data: Block Bootstrap

- Numerous other block bootstrap procedures exist.(See Lahiri, 1999, *Annals of Statisitcs*)
- The performance of the block bootstrap is far from satisfactory, and is very sensitive to block length b.

Notes:

Dependent Data: Sieve Bootstrap

- Step 1: Fit $AR(p)$ to the residuals, where p is chosen based on the AIC or BIC criterion.
  i.e.

$$x_i = \widehat{\alpha}_0 + \widehat{\alpha}_1 x_{t-1} + \ldots + \widehat{\alpha}_p x_{i-p} + \epsilon_i$$

  where $\{x_i\}_{i=1}^n$ are the residuals, $\{\widehat{\alpha}_i\}_{i=0}^p$ are the AR estimates, and $\{\epsilon_i\}$ are the redoduals from the autoregression.

Notes:

Dependent Data: Sieve Bootstrap

- Step 2: Generate the bootstrap samples $\{\epsilon_i^*\}$ form the EDF $\{\epsilon_i\}$.
- Step 3: Generate the bootstrap residuals $\{x_i^*\}$ by

$$x_i^* = \widehat{\alpha}_0 + \widehat{\alpha}_1 x_{t-1}^* + \ldots + \widehat{\alpha}_p x_{i-p}^* + \epsilon_i^*$$

Notes:

Dependent Data: Sieve Bootstrap

- The idea behind the procedures is that the DGP can be represented as an infinite-order autoregression that can be approximated by $AR(p)$.
- The AR order p has to grow at some suitable rate as $n \to \infty$.
- Sieve bootstrap performs better than block bootstrap in many cases.

Notes:

Wild Bootstrap to Deal with Heteroscedasticity

- Consider the regression model

$$y_i = x_i\beta + u_i, u_i = \sigma_i\epsilon_i, E(\epsilon_i^2) = 1$$

where σ_i^2 the error variance, depends of the regressors in an unknown fashion.

Notes:

Wild Bootstrap to Deal with Heteroscedasticity

- The "wild bootstrap" DGP would be

$$y_i^* = x_i\widetilde{\beta} + f(\widetilde{u}_i)\nu_i^*$$

where
$\widetilde{\beta}$ would be restricted or unrestricted estimates, $f(\widetilde{u}_i)$ is a transformation of the i^{th} residual associated with $\widetilde{\beta}$ and ν_i^* is a random variable with mean 0 and variance 1.

Notes:

Wild Bootstrap to Deal with Heteroscedasticity

- A simple choice for $f(.)$ is

$$f(\widetilde{u}_i) = \frac{\widetilde{u}_i}{(1-h_i)^{1/2}}$$

- Many ways to specify ν_i^*. One popular choice is

$$\nu_i^* = \begin{cases} -1 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2 \end{cases}$$

Notes:

Wild Bootstrap to Deal with Heteroscedasticity

- In some respects, the error terms for the wild bootstrap DGP do not resemble those of the true DGP.
- Nevertheless, the wild bootstrap does mimic the essential features of the true DGP well enough for it to be useful in many cases.

Notes:

Bootstrap Inference

- It is important to generate a valid bootstrap distribution under the null for correctly making inference from the data.
- For this purpose, the null hypothesis must be imposed when generating the bootstrap samples.

Notes:

Bootstrap Unit Root Testing

- Unit root (ADF) test as an example:
  The ADF regression

$$\Delta x_t = \beta_0 + \beta_1 x_{t_1} + \sum_{j=1}^{p} \delta_j \Delta x_{t-j} + u_t$$

The hypothesis of interest is

H$_0$: $\beta_1 = 0$ (unit root)

Notes:

Bootstrap Unit Root Testing

- The test employed is the ordinary t statistic whose limit distribution is non-standard.
- The asymptotic approximation works very bad in typical small samples. The bootstrap approach is called for.

Notes:

Bootstrap Unit Root Testing

- The bootstrap procedures are very much the same as previously described, except that the bootstrap sample $\{x_t^*\}$ is generated by

$$\Delta x_t^* = \widetilde{\beta}_0 + \sum_{j=1}^{p} \widetilde{\delta}_j \Delta x_{t-j}^* + u_t^*$$

where the restriction $\beta_1 = 0$ is imposed.

Notes:

The Problem of the Stationarity Tests

- KPSS (1992) documented that their tests are subject to the size problem in the presence of highly persistent process.
- Caner and Kilian (2001) reaffirmed the problem.

Notes:

Construction of the KPSS Tests

- regress y_t against an intercept, or an intercept and time trend, and obtain the residuals, denoted by $\hat{u}_t$.
- compute the partial sum of the residuals, $S_t = \sum_{i=1}^{t} \hat{u}_i$, and estimate the long-run variance of ϵ_t:

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 + 2\frac{1}{T}\sum_{i=1}^{L} w(i,L) \sum_{t=i+1}^{T} \hat{u}_t \hat{u}_{t-i}$$

where $w(i,L) = 1 - i/(1+L)$ is Bartlett kernel, and L is bandwidth.

Notes:

Construction of the KPSS Tests

- compute the statistic:

$$KPSS = T^{-2}\hat{\sigma}^{-2}\sum_{t=1}^{T} S_t^2$$

Notes:

The Dangerous Zone: AR coefficient near 1

- Simulated illustration to replicate findings of Caner and Kilian (2001).
  - DGP: $y_t = \alpha y_{t-1} + e_t$, $e_t \stackrel{iid}{\sim} N(0,1)$.
  - Nominal level: 5%, replications: 5000.

Notes:

Table 1. Size Behavior of the KPSS Test

T	α	$KPSS_\tau(4)$	$KPSS_\tau(8)$	$KPSS_\tau(12)$
300	0.99	0.976	0.882	0.767
	0.98	0.963	0.832	0.698
	0.94	0.846	0.570	0.393
	0.30	0.073	0.062	0.056
600	0.99	0.995	0.957	0.837
	0.98	0.986	0.901	0.774
	0.94	0.847	0.575	0.382
	0.30	0.071	0.056	0.053

Notes:

The Dangerous Zone: AR coefficient near 1

- Size distortion worsens as samples increase, and as AR coefficient approaches to 1.
- For fixed α that is near 1, size problem is reduced by an increase in bandwidth number.

Notes:

The DGP

- DGP: represented by a component model:

$$y_t = r_t + u_t$$
$$u_t = \alpha u_{t-1} + \epsilon_t$$
$$r_t = r_{t-1} + e_t$$

where $c < 0$, $\epsilon_t \overset{iid}{\sim} (0, \sigma_\epsilon^2)$, $e_t \overset{iid}{\sim} (0, \sigma_e^2)$.

- The null of concern

$$H_0 : \sigma_e^2 = 0$$

Notes:

Problems with Bootstrapping Tests for Stationarity

- It is not clear how to impose the null constraint

$$H_0 : \sigma_e^2 = 0$$

when bootstrapping in the context.

- This is more due to the impossibility of untangling the stationary component from the data in components models on which the tests are built.

Notes:

Our Solutions: Using the Reduced Form

- The component model is 2^{nd}-order observationally equivalent in moments to an ARMA(1,1) for the differenced data:

$$\Delta y_t = \alpha \Delta y_{t-1} + (1 - \theta L)\eta_t,$$

where $\eta_t \overset{iid}{\sim} (0, \sigma_\eta^2)$, $\sigma_\eta^2 = \sigma_\epsilon^2/\theta$, and $\theta = \frac{1}{2}\{\frac{\sigma_e^2}{\sigma_\epsilon^2} + 2 - (\frac{\sigma_e^2}{\sigma_\epsilon^2} + 4\frac{\sigma_e^2}{\sigma_\epsilon^2})^{1/2}\}$.

Notes:

Our Solutions: Using the Reduced Form

- Based on the equivalence,

$$H_0 : \sigma_e^2 = 0 \Longleftrightarrow \theta = 1.$$

- The equivalence renders the bootstrapping feasible by imposing $\theta = 1$ in the resampling scheme.

Notes:

The Re-sampling Algorithm

- Step 1: Give a sample $\{y_t\}_{t=1}^T$, fit Δy_t with an ARMA(1,1) model

$$\Delta y_t = \alpha \Delta y_{t-1} + \eta_t - \theta \eta_{t-1}$$

  to obtain $\widehat{\alpha}$, $\widehat{\theta}$ and $\{\widehat{\eta}_t\}_{t=1}^T$.

- Step 2: Draw a bootstrap sample from EDF $\{\widehat{\eta}_t\}_{t=1}^T$, denoted by $\{\eta_t^*\}_{t=1}^T$, with replacement.

Notes:

The Re-sampling Algorithm

- Step 3: Generate a bootstrap sample $\{y_t^*\}_{t=1}^T$ by the following bootstrap DGP:

$$y_t^* = y_{t-1}^* + \widehat{\alpha} \Delta y_{t-1}^* + \eta_t^* - \eta_{t-1}^*$$

- Step 4: Compute KPSS test statistics, based on $\{y_t^*\}_{t=1}^T$.
- Step 5: Repeat step 2 to step 4 NB times, to produce the bootstrap "null" distribution.
- Step 6: Compute the bootstrap C.V. at desired levels.

Notes:

Table 2-1. Simulation Evidence: Size

T	α	$KPSS_\mu(4)$		$KPSS_\mu(8)$		$KPSS_\mu(12)$	
		$asym.$	$boot.$	$asym.$	$boot.$	$asym.$	$boot.$
100	0.98	0.743	0.141	0.569	0.117	0.451	0.097
	0.94	0.588	0.099	0.395	0.086	0.275	0.068
	0.90	0.462	0.097	0.282	0.073	0.188	0.064
	0.80	0.271	0.086	0.151	0.075	0.100	0.064
	0.50	0.102	0.062	0.067	0.061	0.049	0.059
	0.00	0.049	0.061	0.042	0.062	0.036	0.065

Notes:

1. $DGP: y_t = \alpha y_{t-1} + e_t$, $e_t \stackrel{iid}{\sim} N(0,1)$.
2. Nominal significance: 0.05; asym. c.v. 0.463; replications: 5000.
3. NB=100, ND=1000.

Table 2-2. Simulation Evidence: Size

T	α	$KPSS_\mu(4)$		$KPSS_\mu(8)$		$KPSS_\mu(12)$	
		$asym.$	$boot.$	$asym.$	$boot.$	$asym.$	$boot.$
150	0.98	0.845	0.144	0.659	0.123	0.535	0.107
	0.94	0.664	0.107	0.435	0.090	0.311	0.076
	0.90	0.500	0.096	0.292	0.075	0.195	0.062
	0.80	0.272	0.072	0.147	0.058	0.103	0.058
	0.50	0.102	0.059	0.066	0.051	0.052	0.054
	0.00	0.045	0.066	0.043	0.061	0.038	0.062

Notes:

1. $DGP: y_t = \alpha y_{t-1} + e_t$, $e_t \stackrel{iid}{\sim} N(0,1)$.
2. Nominal significance: 0.05; asym. c.v. 0.463; replications: 5000.
3. NB=100, ND=1000.

Table 3-1. Simulation Evidence: Power

T	σ^2_ζ	$KPSS_\mu(4)$		$KPSS_\mu(8)$		$KPSS_\mu(12)$	
		$asym.$	$boot.$	$asym.$	$boot.$	$asym.$	$boot.$
100	1	0.821	0.861	0.690	0.729	0.612	0.652
	0.1	0.759	0.795	0.654	0.676	0.586	0.614
	0.01	0.506	0.540	0.465	0.495	0.426	0.458
	0.001	0.146	0.163	0.137	0.155	0.134	0.159
	0.0001	0.057	0.069	0.057	0.071	0.058	0.074

1. $DGP : y_t = \alpha y_{t-1} + e_t$, $e_t \overset{iid}{\sim} N(0,1)$.
2. Nominal significance: 0.05; asym. c.v. 0.463; replications: 5000.
3. NB=100, ND=1000.

Notes:

Table 3-2. Simulation Evidence: Power

T	σ^2_ζ	$KPSS_\mu(4)$		$KPSS_\mu(8)$		$KPSS_\mu(12)$	
		$asym.$	$boot.$	$asym.$	$boot.$	$asym.$	$boot.$
150	1	0.914	0.928	0.792	0.804	0.710	0.728
	0.1	0.881	0.899	0.765	0.783	0.693	0.708
	0.01	0.681	0.699	0.618	0.633	0.573	0.576
	0.001	0.278	0.309	0.262	0.300	0.248	0.278
	0.0001	0.077	0.100	0.072	0.094	0.074	0.101

1. $DGP : y_t = \alpha y_{t-1} + e_t$, $e_t \overset{iid}{\sim} N(0,1)$.
2. Nominal significance: 0.05; asym. c.v. 0.463; replications: 5000.
3. NB=100, ND=1000.

Notes:

Empirical Application

- Evidence in support of PPP in previous studies was mixed, based on the asymptotic C.V.
- Stronger evidence for PPP in OECD real exchange rates series by the bootstrap tests are observed, accounting for size distortions.

Notes:

Table 4: KPSS and bootstrap KPSS tests for real exchange rate

country	KPSS value	$\hat{\alpha}$	bcv(5%)	bcv(10%)
Austria	$0.439^{*\dagger}$	0.678	0.561	0.438
Belgium	0.224	0.783	0.693	0.535
Canada	0.924^{**}	1.000	2.058	2.055
Denmark	0.276	0.666	0.558	0.417
Finland	0.119	0.767	0.662	0.533
France	0.196	0.689	0.581	0.443
Germany	0.237	0.728	0.659	0.475
Greece	0.355^{*}	1.000	2.058	2.055
Italy	0.367^{*}	0.982	1.801	1.747
Japan	$1.315^{**\ddagger}$	0.748	0.663	0.527
Netherlands	0.221	0.553	0.532	0.419
Norway	0.156	0.607	0.490	0.387
Portugal	0.459^{*}	0.799	0.629	0.495
Spain	0.395^{*}	0.759	0.641	0.513
Sweden	0.296	1.000	2.058	2.053
Switzerland	$0.608^{**\ddagger}$	0.608	0.529	0.441
United Kingdom	0.410^{*}	0.798	0.643	0.493

Notes: All real exchange rates are constructed from IFS CD-ROM data on consumer price and end-of-period US$ exchange rates. Monthly data are for 1973.1—1998 the 5 (10)% significance level, the asymptotic critical value for KPSS test is 0.463 (0.347). $**$ ($*$) denotes a rejection at 5 (10)% level.

Notes:

Bootstrap Inference

- Kuo and Lee (2002), Kuo and Tsong (2002), Kuo and Lee (2002), and Kuo, Lee ans Tsong (2004) all utilize this notion in constructing bootstrap tests for panel unit root, and for equal forecast accuracy.
- Kuo and Tsong (2004) transform the component model on wihch the KPSS is based into a reduced form to have the null hypothesis imposed when generating bootstrap samples.

Notes:

Other Issues

- Some important estimators are not covered, including kernel density estimator, non-parametric regression, GMM estimator, and non-smooth estimator.

Notes: