Analyzing the Light Path in a Spherically Symmetric Medium in Terms of a Mechanical Model

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We analyze the light path in a spherically symmetrical medium, \( n(r) = \sqrt{1 + r_0^2/r^2} \). We find that the ray path may be viewed as a trajectory of a particle for which the mass is velocity dependent. It is found to be \( m = m_0 n^2 = m_0 c^2/v^2 \), where \( m_0 \) is an arbitrary constant mass and \( v \) is the speed of light in the medium. The equation governing the motion is \( \mathbf{F} = \frac{d}{dr} (m \mathbf{r}) \), where \( \mathbf{F} \) is a central force: \( \mathbf{F} = \frac{d}{dr} (m_0 c^2 \ln n(r)) \mathbf{r} \).

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I. INTRODUCTION

It is known that the path of a light ray (abbreviated as the light path) can be described by Fermat’s least time principle [1, 2], and the path of a particle can be described by Hamilton’s least action principle. Since light rays and particles are two quite different things, we take these two principles as separate. However, suppose the light path can be interpreted in terms of a mechanical model, that is, we take the light path \( \mathbf{r}(t) \) as a trajectory of a particle moving under the action of some force. In this way we take the light to be an effective particle, and then we get a trajectory for the particle whose guiding principle is Fermat’s least time principle. The force law derived from this principle is expected to be different from that derived from Hamilton’s principle. It is interesting to compare the dynamics of these two types of particle.

A recent paper [3] discussed the light path with refractive index \( n(r) = n(r) = \sqrt{1 + r_0^2/r^2} \), where \( r_0 \) is a constant. The solution \( r(\theta) \) in this model can be found exactly. Consequently, we may also want to investigate the time behavior of the trajectory, that is \( \mathbf{r}(t) \). We found that we could exactly solve for the time behavior \( t(r) \). The reverse relation \( r(t) \) is difficult to obtain. However, it is interesting to note that, even so, \( \mathbf{r}(t) \) and \( \mathbf{r}(t) \) can still be found exactly. From this, we may then try to establish the force law for this specific model.

The acceleration \( \mathbf{\ddot{r}} \), after calculation, is found to have both \( \mathbf{\dot{r}} \) and \( \mathbf{\dot{\theta}} \) components. This shows that a mechanical model of the Newtonian type is not suitable for describing the path \( \mathbf{\mathbf{\mathbf{r}}}(t) \), since the force \( \mathbf{F} = m \mathbf{\ddot{r}} \) has a \( \theta \) component. This does not reflect the spherical symmetry of the environment. We can then try to express the force law in the form \( \mathbf{F} = \frac{d}{dr}(m \mathbf{\ddot{r}}) = m \mathbf{\dot{r}} + m \mathbf{\ddot{r}} \). This is similar to the force law in relativistic mechanics for which the mass of the particle is velocity dependent. It is surprising to find that, by requiring the force \( \mathbf{F} \) to have only an \( \mathbf{\dot{r}} \) component (in order to reflect the symmetry of the
environment), the mass $m$ can be determined to be $m = m_0 n^2 = m_0 c^2 / v^2$, where $m_0$ is an arbitrary constant representing the mass when $n = 1$. The mass formula is quite different from that used in relativistic mechanics. We discuss the details below.

II. THE TIME BEHAVIOR OF THE TRAJECTORY OF LIGHT

The trajectory of light is determined from Fermat’s principle:

$$\delta \int dt = 0,$$  \hspace{1cm} (1)

where the integration is taken along the true physical path of the light. For a spherically symmetrical medium, $n = n(r)$, all the light paths are plane curves \[1\], therefore we can describe them by $\theta = \theta(r)$, where $\theta$ is the polar angle. Taking $dt = \frac{ds}{v} = \frac{v ds}{c} = \frac{2}{c} \sqrt{dr^2 + r^2 d\theta^2}$ in (1), we have

$$\delta \int n(r) \sqrt{1 + r^2 (d\theta / dr)^2} dr = 0.$$  \hspace{1cm} (2)

The Euler-Lagrange equation derived from (2) is

$$\frac{r^2 n(r) (d\theta / dr)}{\sqrt{1 + r^2 (d\theta / dr)^2}} = r_i,$$  \hspace{1cm} (3)

where the constant $r_i$ represents the impact parameter of the incident light. From (3) we have

$$\frac{d\theta}{dr} = \pm \frac{r_i}{\sqrt{r^4 n^2(r) - r_i^2 r^2}}.$$  \hspace{1cm} (4)

The solution for the light path $\theta(r)$ with $n = \sqrt{1 + r_i^2 / r^2}$ was exactly solved in (3). The trajectories can be classified into three types:

II-1. For the case $r_i = r_0$

The trajectory is $\theta = r_0 / r$. To determine the time behavior, we integrate $dt = -n \sqrt{1 + r^2 (d\theta / dr)^2} dr / c$. The minus sign is to take into account that $dr < 0$ when the particle comes in from infinity. We obtain

$$t = -\frac{r}{c} + \frac{r_0^2}{cr},$$  \hspace{1cm} (5)

where we set $t = 0$, when $r = r_0$. From (5), we get the following results: for $r \to \infty$, we have $t \to -\infty$; for $r \to 0$, we have $t \to \infty$. Thus light is finally trapped to the origin. If we differentiate (5) with respect to $t$, we obtain

$$\dot{r} = \frac{(-c r^2)}{r^2 + r_0^2} = -\frac{c}{n^2}.$$  \hspace{1cm} (6)
From (6), we obtain the following quantities:

\[ \ddot{r} = \frac{2c^2 r_0^2}{n^6 r^3}, \]  
\[ \dot{\theta} = \frac{r_0 c}{n^2 r^2}, \]  
\[ \ddot{\theta} = \frac{2r_0 c^2}{n^6 r^3}. \]

(7)

The quantities \( \ddot{r} = \dot{r}^2 + r \dot{\theta} \) and \( \ddot{\theta} = (\dot{r} - r \dot{\theta}^2) \dot{r} + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \dot{\theta} \) can therefore be determined from (6) and (7). We now have

\[ \ddot{\gamma} r = \frac{c}{n} \left( \frac{1}{n} \ddot{r} + \frac{r_0}{n} \ddot{\theta} \right), \]
\[ \ddot{\gamma} r = \frac{(2 - n^2) r_0^2 c^2}{n^6 r^3} \ddot{r} + \frac{(-2) c^2 r_0^3}{n^6 r^4} \dot{\theta}. \]

(8)

We note that the \( \dot{\theta} \) component of \( \ddot{r} \) is not zero. Thus, if the force law governing the motion is of the Newtonian type: \( \ddot{F} = m \ddot{r} \), then \( \ddot{F} \) is not a central force. This is in contrast to the symmetry of the environment, which has a spherically symmetrical refractive index. We then try to eliminate the \( \dot{\theta} \) component of \( \ddot{F} \). We achieve this by modifying the equation of motion to \( \ddot{F} = \frac{d}{dt}(m \ddot{r}) \), where the mass \( m \) will not be taken as a constant, but rather time dependent. Then

\[ \ddot{F} = m \ddot{r} + m \ddot{r} = \left( \dot{m} \ddot{r} + m \ddot{r} - mr \dot{\theta}^2 \right) \dot{r} + \left( \dot{m} r \dot{\theta} + 2m \ddot{r} + m \dot{r} \right) \dot{\theta}. \]

(9)

Now requiring that the \( \dot{\theta} \) component of (9) be zero, we obtain an equation for \( m \), which we call the mass equation:

\[ \dot{m} r \dot{\theta} + 2m \ddot{r} \dot{\theta} + m r \ddot{\theta} = 0. \]

(10)

From (6) and (7), we note the relation

\[ \ddot{\theta} = \frac{(-2)}{n^2 r^3} \dot{r} \dot{\theta}. \]

(11)

Taking this into the mass equation (10), we obtain

\[ \dot{m} = m \frac{2}{r} \left( \frac{1}{n^2} - 1 \right) \dot{r} = m \frac{(-2) r_0^2}{n^6 r^3} \dot{r}. \]

(12)

From (12), we have \( \frac{dm}{m} = \frac{(-2) r_0^2}{n^6 r^3} dr \). By integrating both sides of this equation, we obtain

\[ m = m_0 n^2, \]

(13)
where $m_0$ is an arbitrary constant, defined to be the mass of the particle when $n = 1$. Thus the mass of the particle is velocity dependent. Formula (13) can be rewritten as

$$mv^2 = m_0 c^2.$$  \hfill (14)

The mass $m$ is larger when the velocity is slower, and smaller when the velocity is faster. This is contrary to that in relativistic mechanics. The force is now a central force and is

$$\vec{F} = (\dot{m}\vec{r} + m\vec{v} - mr\dot{\theta}^2)\hat{r}.$$  \hfill (15)

From (15), we obtain

$$\vec{F} = -m_0 c^2 \frac{r_0^2}{n^2 r^3} \hat{r}.$$  \hfill (16)

### II-2. For the case of $r_i > r_o$

Let $r_m = \sqrt{r_i^2 - r_0^2}$. The trajectory in this case is $r = r_m / \sin(\theta(r_m/r_i))$. Thus $r \geq r_m$. The particle is therefore not trapped at the singularity of the refractive index. The nearest distance to the origin is $r_m$. The time behavior of the light path in the range $-\infty \geq t \geq 0$ is determined to be

$$t = -\frac{1}{c} \sqrt{r^2 - r_m^2} + \frac{r_0^2}{cr_m} \tan^{-1} \left( \frac{\sqrt{r^2 - r_m^2}}{r_m} \right),$$  \hfill (17)

where we have set the integration constant to zero. Thus light begins its journey from $r \to \infty$ at $t \to -\infty$, reaching $r = r_m$ at $t = 0$. The time behavior in the range $0 \geq t \geq \infty$ is obtained by replacing the minus sign with the positive sign. Thus, light goes to $r \to \infty$ at $t \to \infty$, that is, light finally turns outward and escapes. From (17), we get

$$\dot{r} = \frac{-(c^n) \sqrt{r^2 - r_m^2}}{n^2},$$  \hfill \frac{\dot{r}}{c} = \frac{cr_i}{n^2 r^2},$$

$$\dot{\theta} = \frac{2c^2 r_i}{n^6 r^4} \sqrt{r^2 - r_m^2}.$$  \hfill (18)

From (18), we again obtain relation (11). Thus the mass $m$ determined from (10) is of the same value as (13) and (14). The force calculated from (15) is also the same as (16).

### II-3. For the case $r_i < r_0$

Let $r_m = \sqrt{r_0^2 - r_i^2}$. The trajectory is $r = r_m / \sinh(\theta(r_m/r_i))$ \cite{4}. The time behavior of the trajectory can be determined to be

$$t = -\frac{\sqrt{r^2 + r_m^2}}{c} + \frac{r_0^2}{cr_m} \coth^{-1} \left( \frac{\sqrt{r^2 + r_m^2}}{r_m} \right),$$  \hfill (19)
where we set the integration constant to zero. For $r \to \infty$, we have $t \to -\infty$; and for $r \to 0$, we have $t \to \infty$. Thus, light is finally trapped into the singularity of the refractive index.

We now have
\[
\dot{r} = \left(\frac{-c}{n^2} \right) \frac{\sqrt{r^2 + r_m^2}}{r},
\]
\[
\dot{\theta} = \frac{cr_i}{n^2 r^2},
\]
\[
\ddot{\theta} = \frac{2c^2 r_i}{n^6 r^4} \sqrt{r^2 + r_m^2}.
\]

From (20), we note that we still have relation (11), and therefore the mass $m$ determined from (10) is of the same value as (13) and (14). We see, interestingly, that in all three cases, the trajectory of light can be viewed as the path of a particle with mass $m = m_0 n^2$ moving under a central force. The forces calculated from (15) in all three cases all have the same value as (16). We may rewrite the force by noting that $dn/dr = -r_0^2/(nr^3)$. Thus the force can therefore be written as
\[
\vec{F} = m_0 c^2 \frac{1}{n} \frac{dn}{dr} \hat{r} = m_0 c^2 \frac{d}{dr} (\ln n(r)) \hat{r}.
\]

As the force is central, the angular momentum $\vec{L} = \hat{r} \times m \hat{r}$ is conserved. We have $|\vec{L}| = m_0 r_0^2 \hat{r} = m_0 c r_i$. This is equivalent to formula (3). Thus the Euler-Lagrange equation (3) can be interpreted in terms of the mechanical model as the conservation of angular momentum.

It is interesting to see whether this mechanical interpretation of the light path in (13), (14) and (21) can be extended to arbitrary spherically symmetrical mediums.

III. CONCLUSION

We have shown that the light path in a spherically symmetric medium has a mechanical interpretation, that is, the light path is equivalent to the path of a particle with mass $m$ under an external central force. The exact meaning of this mechanical interpretation of the light path has yet to be explored. We give some further discussion.

(1) The concept of a light path is referred to in geometric optics, that is, the case of short wavelengths. The wave property of light is not noticeable in this case, so we can talk about the trajectory of light in the same way as we talk of the trajectory of a particle. This leads us to the particle model of a light ray. The equation governing the motion of a light ray is similar to that used in relativistic mechanics, $\vec{F} = \frac{d}{dt}(m \vec{v})$. However, the mass formula is different, we have $m = m_0 \sqrt{1 - v^2/c^2}$, instead of $m = m_0/\sqrt{1 - v^2/c^2}$. Thus the mechanical model for a light path is different from that in classical mechanics and relativistic mechanics. The light path analysis offers us a new type of particle dynamics. The kinematics of this model needs further investigation.

(2) As we know that the photon is massless, we may wonder why its light path can be seen as being equivalent to the trajectory of a massive particle? However, we note that
a massless photon is for the case where light is propagating in a vacuum. Since the velocity of light in a medium is \( \frac{c}{n} \), the concept of a massless photon is lost when light is propagating in a medium with \( n > 1 \). The effective mass of a photon in a medium is certainly not zero. Here, it seems acceptable to treat the light path in a medium as a trajectory of a massive particle. Also, the dynamics for this massive particle is different from that for normal particles. In particular, the meaning of \( m_0 \) in the mass formula, \( m = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} \), is still unclear, as its value can be arbitrary, and the description of the light path does not depend on the value of \( m_0 \). We do not yet know the meaning of \( m_0 \).

(3) Since the light path follows from Fermat’s principle, and we have given the light path a mechanical interpretation, we have therefore established a mechanical model for which the path of a particle does not follow Hamilton’s principle, but rather Fermat’s principle. In another words, we show that it is possible to establish a new mechanism by which the trajectory of a particle follows the principle of least time. However, it is curious: do we need two different principles for analyzing the trajectories?

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References

[4] Formula (24) in reference [3]; \( r = \sqrt{r_1^2 - r_2^2} \) is a missprint.