Solutions of a Cubic Equation and their Applications in Matrix Analysis

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0. Abstract

The abstract. The solution formula of a cubic equation was found by Italian mathematicians in the 16th century. Modern methods to solve a cubic equation are related with the formula $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ or an elliptic function. The solution formula of a cubic equation provides a fundamental tool to analyse 3×3,5×5 or 6×6 matrices. In my talk, some applications are presented.

Our theme ..... 温故知新

1. a brief review of the Galois theory

Jean-Batiste d'Alembert (1717-1783)

Johann C. F. Gauss (1777-1855)

Consider an n-th order algebraic equation ($n \geq 1$):

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0, \tag{1.1}$$

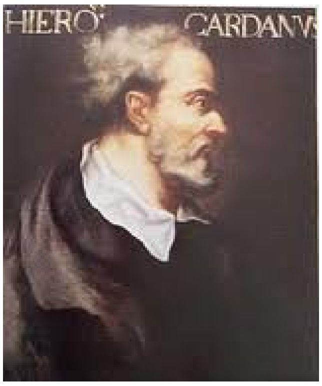
with complex coefficients a_j ($a_0 \neq 0$).

The fundamental theorem of algebra states that every non-constant single-variable polynomial $P(x)$ has one complex root $x_0 \in \mathbb{C}$: $P(x_0) = 0$.

This theorem was proved by a French mathematician d'Alembert in 1746. It was also proved by a German mathematician Gauss in 1799.

But an efficient solution method of an algebraic equation is another problem. Suppose that the coefficients a_j belong to a filed $K \subset \mathbb{C}$. If $n = 1$, the solution x_0 of (1.1) belongs to K. We assume that $n \geq 2$. In 628 AD, an Indian mathematician Brahmagupta gave the first (although not completely general) solution of the quadratic equation $ax^2 + bx = c$ as

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a}.$$



G. Cardano (1501-1576)

N. Fontana (1499/1500-1557)

16th-century Italian mathematian:

Gerolamo Cardano (1501-1576), Niccoló Fontana [Tartaglia] (1449/1500-1557),Scipione del Ferro (1465-1526) provided a solution formula of a cubic equation. Their solution is an algebraic solution using cubic roots $a^{1/3}$ of (complex) numbers a and squre roots $b^{1/2}$ of numbers b.

Ludovico Ferrari (1522-1565) used a bi-quadratic equation to reduce a quartic equation to a cubic equation (Cardano's "Ars Magna" (the Great Art) in 1545).

We wish to express the roots of (1.1) as some functions of the coefficients $a_0, a_1, \dots, a_n$. An algebraic solution of (1.1) corresponds to successive operations of $a \mapsto a^{1/m}$ for some $m = 2, 3, \dots$ and the 4 operatrions $+, -, \times, /$. By using the field theory, this is equivalent to express every root α of (1.1) as an element of the field $L = L_k$ in the series of the fields

$$L_0 = K \subset L_1 \subset L_2 \subset \cdots \subset L_k = L$$

for which $L_{j+1} = L_j(a_j^{1/m_j})$ for some $a_j \in L_j$, $m_j \geq 2$ ($j = 0, 1, \dots, k-1$).

We consider a primitive m-th root α of $a \in K$. Another m-th root β of a satisfies $(\beta/\alpha)^m = 1$. Corresponding to this, the field $L = K(\alpha, \alpha\omega, \alpha\omega^2, \dots, \alpha\omega^{n-1})$ for $\omega = \exp(i2\pi/n)$ is obtaineed as the folowing:

$$L_0 = K, L_1 = K(\omega), L = L_2 = L_1(\alpha).$$

For a long time, mathematicians were trying to find an algebraic solution of a quintic equation.

A French mathematician Évariste Galois (1811-1832) provided a group theoretic approach to an algebraic solution of the equation.

Suppose that a polynomial $P(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$ $(n \geq 2)$ has rational coefficients and it is irreducible in $\mathbb{Q}[x] = K[x]$. Hence the $P(x) = 0$ has n distinct solutions $\alpha_1, \alpha_2, \ldots, \alpha_n$.

We consider the field $L = K(\alpha_1, \alpha_2, \ldots, \alpha_n)$. For an arbitrary root α_j of $P(x)$, the elements $1, \alpha_j, \alpha_{j-1}^2, \ldots, \alpha_j^{n-1}$ form a basis of the K-vector space L:

$$L = \{c_0 + c_1\alpha_j + c_2\alpha_j^2 + \ldots + c_{n-1}\alpha_j^{n-1} : c_0, c_1, \ldots, c_{n-1} \in K\}.$$

A K-linear map ϕ of L onto itself satisfying

$$\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta), \quad \phi(\alpha\beta) = \phi(\alpha)\phi(\beta), \quad \phi(1) = 1$$

induces a permutation $\sigma \in S_n$ for which $\phi(\alpha_j) = \alpha_{\sigma(j)}$. Hence such K-automorphisms of L form a finite group G. This group G is viewed as a subgroup of S_n. It is called the Galois group of the polynomial $P(x)$.

Theorem 1.1 [Galois] The equation $P(x) = 0$ has an algebraic solution if and only if its Galois group G is solvable as a finite group, that is, there is a series of subgroups of G:

$$G_{k+1} = \{e\} \subset G_k \subset G_{k-1} \subset \cdots \subset G_1 \subset G_0 = G$$

for which G_{j+1} is a normal subgroup of G_j and the quotient group G_j/G_{j+1} is a commutative group [abelian group] ($j = 0, 1, \ldots, k$).

In the above, if G_j/G_{j+1} is a finite abelian group, then the group G_j/G_{j+1} is the product of some cyclic groups and hence there are some intermediate subgroups $G_{j+1} = H_{p+1} \subset H_p \subset H_{p-1} \subset \cdots \subset H_0 = G_j$ for which $H_j + 1$ is a normal subgroup of H_j and the quotient group H_j/H_{j+1} is a cyclic group ($j = 0, 1, \ldots, p$).

[Sketch of the proof of the theorem]

Suppose that H is a subgroup of G. Then the set

$$M = L(H) = \{\alpha \in L : \phi(\alpha) = \alpha \, for \, \phi \in H\}$$

is a field satisfying $K \subset M \subset L$. Conversely, if M is a field satisfying $K \subset M \subset L$, then the set

$$H = G(M) = \{\phi \in G : \phi(\alpha) = \alpha \,\text{for}\, \alpha \in M\}$$

is a subgroup of G.

Moreover, the equation $L(G(M)) = M$ holds for a field satisfying $K \subset M \subset L$. The equation $G(L(H)) = H$ also holds for a subgroup H of G. For a subgroup H of G, let $M = L(H)$.

The restriction of every $\phi \in G$ to M is a K-automorphism of M if and only if H is a normal subgroup of G.

The equation $P(x) = 0$ is solved by an algebraic method if and only if the finite group $G = G_0$ has a series of subgroups

$$G_{k+1} = \{e\} \subset G_k \subset G_{k-1} \subset \cdots \subset G_1 \subset G_0 = G$$

for which G_{j+1} is a normal subgroup of G_j ($j = 0, 1, 2, \ldots, k$) and the quotient group G_j/G_{j+1} is a cyclic group ($j = 0, 1, \ldots, k$). □

In the case $P(x) = x^5 - 10x + 5$, the group G is isomorphic to the symmetric group S_5. The group S_5 has a unique non-trivial normal subgroup A_5, the alternating group of degree 5. The group A_5 is a simple group, that is, a normal subgroup N of A_5 satisfying $N \neq G$ consists of one element. So the equation $x^5 - 10x + 5 = 0$ is not solved by an algebraic method.

In the next section, we shall treat cubic equations.

2. Cardano formula and the triple angle formulas

We shall treat a cubic equation

$$x^3 - \frac{3}{4}x - \frac{\cos\theta}{4} = 0, \tag{2.1}$$

Its solutions are given by

$$\frac{1}{2}(\cos\theta + i\sin\theta)^{1/3} + \frac{1}{2}(\cos\theta - i\sin\theta)^{1/3} = \cos(\frac{\theta}{3} + \epsilon\frac{2\pi}{3})$$

($\epsilon = 0, +1, -1$).

This formula is generalized as the following: The solutions of the cubic equation

$$x^3 + px + q = 0, \tag{2.2}$$

are given by

$$x = u + v, \quad x = \omega u + \omega^2 v, \quad x = \omega^2 u + \omega v$$

where $\omega = (-1 + \sqrt{3}i)/2$ and

$$u = [-\frac{q}{2} + \{\frac{q^2}{4} + \frac{p^3}{27}\}^{1/2}]^{1/3}, \quad v = [-\frac{q}{2} - \{\frac{q^2}{4} + \frac{p^3}{27}\}^{1/2}]^{1/3}$$

with $uv = -p/3$.

Supose that $x^3 + px + q = 0$ is a cubic equation with real coefficients p, q with $p \neq 0$.

If $4p^3 + 27q^2 < 0$, then the polynomial $x^3 + px + q$ has 3 distinct real roots. The 3 roots are obtained by applying the formula

$$4x^3 - 3R^2x - R^3\cos(3\theta) = 4(x - R\cos\theta)(x - R\cos(\theta + 2\pi/3))(x - R\cos(\theta - 2\pi/3)), \tag{2.3}$$

($R > 0, \theta \in \mathbb{R}$).

If $4p^3 + 27q^2 > 0$, then the polynomial $x^3 + px + q$ has one real root and a pair of imaginary roots. The coefficient p may be positive, 0 or negative. Firstly we assume that $p < 0$. In this case, the 3 roots are obtained by applying the formulas:

$$4x^3 - 3R^2x - R^3\cosh(3t) = 4(x - R\cosh t)(x - R\cosh(t + 2\pi i/3))(x - R\cosh(t - 2\pi i/3))$$

$$= 4(x - R\cosh t)(x - [-\frac{1}{2}\sinh t + \frac{i\sqrt{3}}{2}\cosh t])(x - [-\frac{1}{2}\sinh t - \frac{i\sqrt{3}}{2}\cosh t]).$$

Secondly we treat the case $p > 0$. In this case the 3 roots are obtained by applying the formula

$$4x^3+3R^2x-R^3\sinh(3t) = 4(x-R\sinh t)(x-R\sinh(t+2\pi i/3))(x-R\sinh(t-2\pi i/3)),$$

$(R > 0, t \in \mathbb{R})$.

If p, q are complex coefficients with $p \neq 0$, then $4x^3 - 3x - \cos\theta = 0$ is viewed as a standard form of a cubic equation by taking a suitable $\theta \in \mathbb{C}$.

The Cardano formula is essentialy reduced to the triple angle formulas

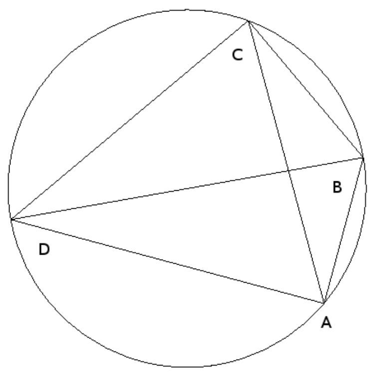
$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta, \quad \cosh(3t) = 4\cosh^3 t - 3\cosh t,$$

$$\sinh(3t) = 4\sinh^3 t + 3\sinh t.$$

At least the formula $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ is deduced from the compound angle formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$



The essence of the compound angle fomula was mentioned by C. Ptolemaeus (ca. AD 90- ca. 168) in his theorem on a convex quadrilateral $ABCD$ inscribed in a circle : The sum of the product of the two pairs of opposite sides equals the product of tho diagonal of the quadrilateral:

$$\overline{AD}\cdot\overline{BC} + AB\cdot\overline{CD} = \overline{AC}\cdot\overline{BD}.$$

Of course Ptolemaeus did not know imaginary numbers.

The most significant contribution of 16th-century Italian mathematicians to mankind would to the beginning of systematic use of imaginary numbers.

A method to solve a cubic equation of the Weierstrass canonical form

$$4x^3 - g_2 x - g_3 = 0,$$

with $\Delta = g_2^3 - 27g_3^2 \neq 0$.

Weierstrass studied the doubly periodic function in a complex variable $y = \mathfrak{p}(x) = \mathfrak{p}(x; g_2, g_3)$ satisfying

$$(\frac{d\mathfrak{p}}{dx})^2 = 4\mathfrak{p}(x)^3 - g_2\mathfrak{p}(x)^2 - g_3.$$

He considered a solution $\mathfrak{p}(x)$ which has a pole of order 2 at $x = 0$. This function $\mathfrak{p}$ has 2 fundamental periods 2ω and $2\omega'$ which are linearly indepent over $\mathbb{R}$. By using this functions $\mathfrak{p}$ and its half-periods ω, ω', the cubic polynomial is factorized as

$$4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

for

$$e_1 = \mathfrak{p}(\omega; g_2, g_3), \quad e_3 = \mathfrak{p}(\omega'; g_2, g_3), \quad e_2 = \mathfrak{p}(\omega + \omega'; g_2, g_3).$$

The software "Mathematica"(Wolfram Research) has a command "Weierstrass HalfPeriods". We can apply this command to obtain the half-periods ω, ω' numericallly. But I am afraid that this command is dependent on the numerical solutions of the cubic equation $4x^3 - g_2 x - g_3 = 0$.

3. The numerical range of the companion matrix of a monic polynomial

The numerical range $W(A)$ of an $n \times n$ complex matrix A is defined as

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}. \tag{3.1}$$

This set $W(A)$ contains the spectrum $\sigma(A)$ of A. The set $W(A)$ is deeply related with the curve $F(1, x, y) = 0$ defined by

$$F(t, x, y) = \det(tI_n + x/2(A + A^*) - yi/2(A - A^*)). \tag{3.2}$$

Kippenhahn proved that the set $W(A)$ is the convex hull of the real affine part of the dual curve of the curve $F(t, x, y) = 0$. For a monic polynomial

$$P(t) = t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_0,$$

there is a standard matrix A satisfying $\det(tI_n - A) = P(t)$. Such a matrix A is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & \ldots & \ldots & \ldots & -a_1 \end{pmatrix}.$$

This matrix is called the companion matrix of $P(t)$. It is very useful in many branches of mathematics. We consider the derivative $P'(t)$ of $P(t)$. Since it is not monic for $n \geq 2$, we normalize as

$$\frac{1}{n} P'(t) = t^{n-1} + \frac{(n-1)a_1}{n} t^{n-2} + \ldots + \frac{a_{n-1}}{n}.$$

The following inclusion holds and it is known as Lucas's theorem:

$$\{t \in \mathbb{C} : P'(t) = 0\} \subset \mathrm{Conv}(\{t \in \mathbb{C} : P(t) = 0\}).$$

In 2011, Professor M. T. Chien posed the following question: C is the companion matrix $P(t)$ and C' is the companion matrix of $(1/n)P'(t)$.

$$W(C') \subset W(C)\,?$$

He provided a counter-example $P(t) = t^3 - 1/2t^2 + t$. For a real cubic polynomial $P(t) = t^3 + a_2 t^2 + a_1 t + a_0$, he provided the necessary and the sufficient condition for $W(C') \subset W(C)$.

Theorem 3.1[M. T. Chien, 2011] Let $p(x) = x^3 + a_2x^2 + a_1x + a_0$ be a real polynomial, and C and C' be respectively the companion matrices of $p(x)$ and $(1/3)p'(x)$. Then $W(C') \subset W(C)$ if and only if one of the following conditions is satisfied.

$$\text{(i)}3a_0^2 + 2a_1^2 \geq 3.$$

$$\text{(ii)}3a_0^2 + 2a_1^2 < 3, \text{and}$$

$$6\{-9a_0a_1 + 3a_0^2a_2 + a_2(-3 - 3a_1 + 3a_1^2 + 2a_2^2)\}\cos\theta$$
$$-\{9 + 9a_0^2 + 8a_1^2 + 4a_2^2 + 4(a_2^2 - 3a_1)\cos 2\theta\}$$
$$\times[9 + a_1^2 + 2a_2^2 + (2a_2^2 - 6a_1)\cos 2\theta]^{1/2}$$
$$+(54a_0 - 18a_1a_2 + 4a_2^3)\cos 3\theta \leq 0$$

for all $0 \leq \theta < 2\pi$.

Especially we are interested in the situation $W(C') \subset W(C)$ and the boundary of $W(C')$ and the boundary of $W(C)$ have a common point. We consider a 1-parameter family of polynomials

$$P(t : \lambda) = t^3 - \frac{1}{2}\lambda t^2 + \lambda t.$$

We examine the condition for the boundaries of $W(C_\lambda)$ and $W(C'_\lambda)$ have a common point on the real line $y = 0$. The condition is given by

$$79\lambda^6 - 684\lambda^5 + 2862\lambda^4 - 4860\lambda^3 + 4293\lambda^2 - 2430\lambda + 729 = 0.$$

Its numerical real solutions are given by $\lambda = 0.853987$ and $\lambda = 1.06638$. For theses values, the critical situation is realized.

4. Application in Matrix Analysis

Let $S = S(a_1, a_2, \ldots, a_n)$ be a matrix defined as

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & a_{n-1} \\ a_n & 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix},$$

where $a_1, a_2, \ldots, a_n$ are arbitrary complex numbers. We introduce a ternary form by

$$F_S(t, x, y) = \det(tI_n + x\Re(S) + y\Im(S)),$$

where $\Re(S) = (S + S^*)/2$, $\Im(S) = (S - S^*)/(2i)$.

We shall apply the Cardano formula to analyse 7×7 weighted shift matrices. We asssume that $a_j > 0$. It is known that the equation

$$F_S(t, \cos\theta, \sin\theta) = t^n + \sum_{k=1}^{m} b_k t^{n-2k} + \frac{(-1)^{n-1}}{2^{n-1}} (\prod_{j=1}^{n} a_j) \cos(n\theta), \qquad (4.1)$$

($n = 2m$ or $n = 2m + 1$) holds for some real coefficients $b_1, \ldots, b_m$ dependinging on a_j. Based on (4.1) for $n = 7$ and the trigonometric identity

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta,$$

we consider a real ternary form

$$\begin{aligned} G(t, x, y) = {} & t^7 + c_1 t^5(x^2 + y^2) + c_2 t^3(x^2 + y^2)^2 + c_3 t(x^2 + y^2)^3 \\ & + c_4(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6), \qquad (4.2) \end{aligned}$$

We assume that $(t, x, y) = (1, 1, 0), (1, g, 0)$ are singular points of the curve $G(t, x, y) = 0$ for some real constant $g \neq 1, 0, -1$. To determine a matrix representation of the form $G(t, x, y)$, we introduce some constants.

$$\begin{aligned} g_1 &= \cos(\pi/7)/\cos(5\pi/7) \sim -1.44504, \\ g_2 &= \sin(\pi/14)/\sin(5\pi/14) \sim 0.24698, \\ g_3 &= 1 - 4\cos^2(\pi/14) \sim -2.80194, \end{aligned}$$

which are the roots of the cubic equation $g^3 + 4g^2 + 3g - 1 = 0$. We introduce one more value $g_4 = -2\cos(\pi/5) - 1 = (-3 - \sqrt{5})/2 \sim -2.61803$, a solution of the quadratic equation $g^2 + 3g + 1 = 0$.

We assume that $(t, x, y) = (1, 1, 0), (1, g, 0)$ are singular points of the curve $G(t, x, y) = 0$ for some real constant $g \neq 1, 0, -1$. This is equivalent to the following system equations:

$$
\begin{aligned}
&c_1 + c_2 + c_3 + c_4 + 1 = 0, \\
&5c_1 + 3c_2 + c_3 + 7 = 0, \\
&c_1 g^2 + 3c_2 g^4 + 5c_3 g^6 + 6c_4 g^7 - 1 = 0, \\
&5c_1 g^2 + 3c_2 g^4 + c_3 g^6 + 7 = 0.
\end{aligned}
$$

Solve the system of these equations, the coefficients c_j are expressed as rational functions in g.

Proposition Let $G(t, x, y)$ be the real ternary form of order 7 satisfying $G(1, 0, 0) = 1$ and $G(t, \cos(2\pi/7)x - \sin(2\pi/7)y, \sin(2\pi/7)x + \cos(2\pi/7)y) = G(t, x, y)$. Supose thar the curve $G(t, x, y) = 0$ has singular points at $(1, 1, 0)$ and $(1, g, 0)$ ($g \neq 1, g \neq -1$). Then the following holds:

(I) If $0 < g < 1$ and $G(t, x, y)$ is hyperbolic with respect to $(1, 0, 0)$ then $0 < g \leq g_2$.

(II) If $-\infty < g < -1$, $g \neq g_4$, and $G(t, x, y)$ is hyperbolic with respect to $(1, 0, 0)$ then $g_3 \leq g < g_4$ or $g_1 \leq g < -1$.

We shall apply the Cardano formula to obtain the following theoem.

Theorem 4.1 Let $G(t, x, y)$ be the real ternary form of order 7 mentioned in the above proposition for some $0 < g < g_2$, $g_3 < g < g_4$ or $g_1 < g < -1$. Then the form $G(t, x, y)$ is realized as $F_S(t, x, y)$ for some $S = S(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_3, \lambda_2, \lambda_1)$ with positive weights.

Proof. We set $\Lambda_j = \lambda_j^2$. Then the necessary and sufficient conditions for Λ_j are given by

$$
P_1 \equiv g^2(g^2 + 3g + 1)(2\Lambda_1 + 2\Lambda_2 + 2\Lambda_3 + \Lambda_4) - 4(2g^4 + 6g^3 + 5g^2 + 6g + 2) = 0,
$$

$$
\begin{aligned}
P_2 \equiv\ & g^4(g^2 + 3g + 1)(2\Lambda_1\Lambda_2 + 2\Lambda_1\Lambda_4 + 2\Lambda_2\Lambda_3 + 2\Lambda_2\Lambda_4 + \Lambda_2^2 + \Lambda_3^2 + 4\Lambda_1\Lambda_3) \\
& -16(g^6 + 3g^5 + 7g^4 + 13g^3 + 7g^2 + 3g + 1) = 0,
\end{aligned}
$$

$$
P_3 \equiv g^4(g^2 + 3g + 1)(2\Lambda_1\Lambda_2\Lambda_3 + 2\Lambda_1\Lambda_2\Lambda_4 + 2\Lambda_1\Lambda_3^2 + \Lambda_2^2\Lambda_4)
$$

$$-64(3g^4+3g^3+11g^2+9g+3)=0,$$

$$P_4 \equiv g^8(g^2+3g+1)^2\Lambda_1^2\Lambda_2^2\Lambda_3^2\Lambda_4 - 16384(g+1)^6 = 0.$$

We eliminate Λ_3, Λ_4 from these equations. The variables Λ_3, Λ_4 are expessed as rational functions of Λ_1, Λ_4 and g. The main equations are given by a cubic equation in Λ_1 and a cubic equation in Λ_2 . Suitable combinations of solutions of $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ of the main equations satisfy the desired conditions of the theorems.

We introduce the changing variables

$$\Lambda_1 = \frac{2(g^4+3g^3+3g^2+3g+1)}{3g^2(g^2+3g+1)} + \tilde{\Lambda}_1, \quad \Lambda_2 = \frac{8(g^2+g+1)}{3g^2} + \tilde{\Lambda}_2.$$

The first of the main equations is given by

$$4\tilde{\Lambda}_1^3 - 3 \times \frac{16Q(g)}{9g^4(g^2+3g+1)^2}\tilde{\Lambda}_1 - \frac{64P(g)}{27g^6(g^2+3g+1)^3} = 0,$$

where

$$P(g) = g^{12}+9g^{11}+18g^{10}-108g^9-735g^8-1908g^7-2583g^6-1908g^5-735g^4$$

$$-108g^3+18g^2+9g+1,$$

$$Q(g) = g^8+6g^7+3g^6-36g^5-67g^4-36g^3+3g^2+6g+1.$$

The second of the man equations is given by

$$4\tilde{\Lambda}_2^3 - 3\times\frac{64(g^4+5g^3+9g^2+5g+1)}{9g^4}\tilde{\Lambda}_2 - \frac{512(g^6-6g^5-48g^4-83g^3-48g^2-6g+1)}{27g^6} = 0$$

By the Cardano formula, the roots of the main equations are parametrized as if $g_1 < g < -1$:

$$\tilde{\Lambda}_1^{(j)} = \frac{4\sqrt{Q(g)}}{3g^2(g^2+3g+1)}\cos(\frac{\Theta(g)}{3} + \frac{2j}{3}\pi),$$

where

$$\Theta(g) = \mathrm{Arc}\cos(\frac{P(g)}{Q(g)^{3/2}}) \in (-\pi, \pi),$$

and

$$\tilde{\Lambda}_2^{(j)} = \frac{8\sqrt{g^4+5g^3+9g^2+5g+1}}{3g^2}\cos(\frac{\Omega(g)}{3} - \frac{2j-2}{3}\pi),$$

where

$$\Omega(g) = \mathrm{Arc}\,\mathrm{Cos}(-\frac{g^6-6g^5-48g^4-83g^3-48g^2-6g+1}{(g^4+5g^3+9g^2+5g+1)^{3/2}}).$$

In the case $0 < g < g_2$ or $g_3 < g < g_4$ or $g_1 < g < -1$, we have similar but slightly different expressions. By choosing suitable combinations of $\tilde{\Lambda_1}^{(j)}$ and $\tilde{\Lambda_2}^{(k)}$, we can get the desired weights. □

参考文献

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Thank you for your attention!