Uniqueness Problem on Numerical Ranges of 3×3 Companion Matrices

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Definition

A : $n \times n$ complex matrix

$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$: numerical range of A

$$x = [x_1 ... x_n]^T, \; y = [y_1 ... y_n]^T$$

$$\Rightarrow \langle x, y\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$$

Properties: $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$

- $W(A)$ nonempty compact subset of $\mathbb{C}$
- U unitary $\Rightarrow W(U^*AU) = W(A)$

$$\begin{aligned} W(U^*AU) &= \{\langle U^*AUx, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\} \\ &= \{\langle AUx, Ux\rangle : x \in \mathbb{C}^n, \|x\| = 1\} \\ &= \{\langle Ay, y\rangle : y \in \mathbb{C}^n, \|y\| = 1\} \\ &= W(A) \end{aligned}$$

- $A \cong B$ if $U^*AU = B$, where U unitary.

$\Rightarrow W(A) = W(B)$

Properties: $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$

- $W(aA + bI) = aW(A) + \{b\} \quad \forall\ a\ , b \in \mathbb{C}$
- $W(\mathrm{Re}A) = \mathrm{Re}W(A), \quad W(\mathrm{Im}A) = \mathrm{Im}W(A)$

  $\mathrm{Re}A = (A + A^*)/2, \quad \mathrm{Im}A = (A - A^*)/(2i)$
- A 2×2 with $\sigma(A) = \{a, b\}$

  $\Rightarrow W(A) =$ an elliptic disc with foci a and b.
- A $n \times n$, $W(A) =$ an elliptic disc with foci a and b

  $\Rightarrow \{a, b\} \subseteq \sigma(A)$
- Toeplitz–Hausdorff (1918–19): $W(A)$ convex

Definition (Companion Matrix)

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a complex polynomial. The companion matrix of $p(z)$ is

$$\mathrm{M(p)} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}.$$

NOTE.

- The characteristic polynomial of $M(p) = \mathrm{p(z)}$.
- All $a_j = 0$, $M(p)$ is the Jordan block J_n.

History on Uniqueness Problem

Theorem (Gau & Wu 2004)

Let A and B be 3×3 companion matrices. If $W(A) = W(B)$ is NOT a non-circular elliptic disc, then $A = B$.

Theorem (Gau 2010)

If A and B are $n \times n$ reducible companion matrices with $W(A) = W(B)$, then $A = B$.

NOTE. A : reducible matrix if $A \cong \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

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NOTE. A : reducible matrix if $A \cong \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

Example (Gau & Wu 2004)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sqrt{3}i & 4 & (\sqrt{3}/4)i \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}i & 4 & -(\sqrt{3}/4)i \end{bmatrix}$$

Then

- $W(A) = W(B) =$ elliptic disc with foci 2 and -2.
- $\sigma(A) = \{2, -2, (\sqrt{3}/4)i\}$ and $\sigma(B) = \{2, -2, -(\sqrt{3}/4)i\}$
  $\Rightarrow A \neq B$

Problem

Let A and B irreducible 3×3 companion matrices such that

$W(A) = W(B)$ is an elliptic disc with foci z_1 and z_2.

What can be said on A and B?

Obviously, $\sigma(A) = \{z_1, z_2, a\}$ and $\sigma(B) = \{z_1, z_2, b\}$.

How to determine such eigenvalues a and b?

Definition

1. $\Omega \equiv \{(z_1, z_2) : z_1 = a\omega_1, z_2 = a\omega_2\}$, where $\omega_j^3 = 1$, $j = 1, 2, 3$.
2. $s \equiv z_1 + z_2$ and $t \equiv z_1 z_2$.

Theorem (Existence Theorem, Modification of Calbeck 2008)

Given $z_1, z_2 \in \mathbb{C}$. $\exists$ 3×3 irreducible companion matrix A with $\sigma(A) = \{z_1, z_2, z_3\} \ni W(A) =$ elliptic disc with foci z_1 and z_2 if and only if

1. $(z_1, z_2) \in \Omega$, $z_3 = -(a/(|a|^2 + 1))\omega_3$.
2. $(z_1, z_2) \notin \Omega$, z_3 *satisfies*

$$\bar{z}((|t|^2 + |s|^2)z^2 + sz - t) + ((\bar{t}s + \bar{s})z^2 + z - s) = 0.$$

NOTE.

- $s^2 = t \Leftrightarrow (z_1, z_2) \in \Omega$.

Definition (Existence Equation, $s = z_1 + z_2, t = z_1 z_2$)

$$\bar{z}((|t|^2 + |s|^2)z^2 + sz - t) + ((\bar{t}s + \bar{s})z^2 + z - s) = 0$$

NOTE.

- The equation has at least one solution. (Calbeck 2008)
- $\bar{z}p(z) + q(z) = 0 \Rightarrow p(z)^2[z\bar{p}(-\frac{q(z)}{p(z)}) + \bar{q}(-\frac{q(z)}{p(z)})] = 0$

  5th degree polynomial in z (no $\bar{z}$'s).
- $\bar{z}p(z) + q(z) = 0 \Rightarrow \# \leq 5n - 5$, where $n =$deg$(p, q) > 1$.

  (Khavinson & Neumann (2006))
- $\#n(z_1, z_2) \leq 3$. (Calbeck's Problem)

Recall that $s = z_1 + z_2$ and $t = z_1 z_2$ & $s^2 = t \Leftrightarrow (z_1, z_2) \in \Omega$.

Corollary (s, t are real. Generalization of Calbeck 2008)

Let $z_1, z_2 \in \mathbb{C}, s, t \in \mathbb{R}$, $\delta \equiv s^2 + 4(1+t)t^2$. $\exists$ 3×3 irreducible companion matrix A with $\sigma(A) = \{z_1, z_2, z_3\} \ni W(A) =$ elliptic disc with foci z_1 and z_2 if and only if

1. $s^2 = t$: $z_3 = s/(t+1)$.
2. $t \geq -1, s^2 \neq t$, *or* $t < -1, \delta \geq 0$: z_3 *is a* ***real*** *solution of*

$$(t^2 + s^2)z^3 + s(t+2)z^2 + (1-t)z - s = 0. \qquad (1)$$

3. $\delta < 0$: *either* z_3 *is the* ***only one real*** *solution of (1)* *or* $z_3 = (s \pm \sqrt{\delta})/(2t^2)$.

Proof. $P(z) \equiv \bar{z}p(z) + q(z) = 0$. Consider $P(z) - \overline{P(z)} = \cdots$.

NOTE. Let A and B be 3×3 companion matrices. Then $W(A) = W(B) =$ elliptic disc $\Leftrightarrow d_A = d_B$, where d_A and d_B are denoted by the minor axis of length of $W(A)$ and $W(B)$, resp..

Theorem (Identical Elliptic Numerical Ranges)

Let A and B be 3×3 irreducible companion matrices with eigenvalues z_1, z_2, a and z_1, z_2, b, resp., $\ni$ their numerical ranges are elliptic discs with foci z_1 and z_2. Then $W(A) = W(B)$ if and only if either

(a) $ab = 0$, $A = B$, or

(b) $ab \neq 0$,

$$\frac{t\bar{a} + s}{a} + t\overline{sa} = \frac{t\bar{b} + s}{b} + t\overline{sb} \in \mathbb{R}$$

Definition (Equality Equation)

$$\frac{t\bar{a}+s}{a}+t\overline{sa}=\frac{t\bar{b}+s}{b}+t\overline{sb}\in\mathbb{R}$$

Corollary (s, t are real)

Let $z_1, z_2 \in \mathbb{C}$, $s, t \in \mathbb{R}$, $\delta \equiv s^2 + 4(1+t)t^2$. $\exists$ *two distinct* 3×3 *irreducible companion matrices* A *and* B *with eigenvalues* z_1, z_2, a *and* $z_1, z_2, b \ni W(A) = W(B) =$ *elliptic disc with foci* z_1 *and* z_2 *if and only if either* $s = 0, t > 1$ *or* $\delta < 0$. *In this case,* $a \neq b$ *and either* $a, b = (\pm\sqrt{t-1})/t$ *or* $a, b = (s \pm \sqrt{\delta})/(2t^2)$.

NOTE. The previous example gives $z_1 = 2$ and $z_2 = -2$.

$$\Rightarrow \delta = -192 < 0.$$

Lemma

A $n \times n$ companion matrix $\Rightarrow e^{i\theta}A \cong M(p_\theta)$ for each rotation $e^{i\theta}$.

Proof. Let $U = \text{diag}\,(e^{i\theta}, 1, e^{-i\theta}, ..., e^{-i(n-2)\theta})$, unitary.

Compute $U^*(e^{i\theta}A)U$ is a companion matrix. ■

NOTE. Check Dimension: $s = z_1 + z_2$ and $t = z_1 z_2$

(Existence Equation)

$$\bar{z}((|t|^2 + |s|^2)z^2 + sz - t) + ((\bar{t}s + \bar{s})z^2 + z - s) = 0,$$

(Equality Equation)

$$\frac{t\bar{a} + s}{a} + t\overline{sa} = \frac{t\bar{b} + s}{b} + t\overline{sb} \in \mathbb{R}$$

Definition

$S_1 = \{(z_1, z_2) : z_1, z_2 \text{ and } 0 \text{ are co-linear or } |z_1| = |z_2|\}$

$S_2 = \{(z_1, z_2) : z_1, z_2 \text{ and } 0 \text{ are not co-linear } \& \ |z_1| \neq |z_2|\}$

Lemma

$\exists$ a rotation $e^{i\theta} \ni e^{i\theta}s$ & $e^{2i\theta}t$ are real $\Leftrightarrow (z_1, z_2) \in S_1$.

NOTE.

- $(z_1, z_2) \in S_1 \overset{e^{i\theta}}{\Longleftrightarrow} s, t \in \mathbb{R}$
- $\begin{cases} s = 0, t > 1 \\ z_3 = (\pm\sqrt{t-1})/t \end{cases} \overset{e^{i\pi/2}}{\Longrightarrow} \begin{cases} \delta = s^2 + 4(1+t)t^2 < 0 \\ z_3 = (s \pm \sqrt{\delta})/(2t^2) \end{cases}$
- $(z_1, z_2) \in S_2 \Rightarrow \exists \ e^{i\theta} \ni s \in \mathbb{R}$ and $t \notin \mathbb{R}$.

Problem (Calbeck 2008)

1. $\#n(z_1, z_2) \leq 3$.
2. $|z_3| \leq 1/2$.

NOTE.

- If $(z_1, z_2) \in S_1$, then Calbeck's problems are solved.
- If $(z_1, z_2) \in S_1$, then the uniqueness problem is solved.

Lemma

Let A and B be 3×3 irreducible companion matrices. If $W(A) = W(B) =$ elliptic disc with foci z_1 and z_2, where $(z_1, z_2) \in S_2$, then $A = B$.

NOTE. The difficulty is to handle the existence equation and equality equation simultaneous.

Let $s \in \mathbb{R}$, $t = u + iv$ and $r \equiv \frac{t\bar{a}+s}{a} + ts\bar{a} = \frac{t\bar{b}+s}{b} + ts\bar{b}$.

If $u^2 + v^2 + ur = 0$, then

$$y^2 + \frac{v}{su}y + \frac{s^2+r}{r^2} = 0$$

$$y^2 + \frac{2sv}{u^2+v^2+s^2}y + \frac{s^4+(u^2+v^2)s^2+2s^2ur+2s^2r-r^3+r^2}{r^2(u^2+v^2+s^2)} = 0,$$

where $a = s/r + iy_1$ and $b = s/r + iy_2$.

Corollary

Let A and B be 3×3 *companion matrices with* $\sigma = \{z_1, z_2, a\}$ *and* $\sigma\{z_1, z_2, b\}$*, resp.,* $\ni W(A) = W(B) =$ *elliptic disc with foci* z_1, z_2*. Then* $A \neq B$ *if and only if*

1. $0 \in (z_1, z_2)$ *and*
2. $D \equiv (|z_1| - |z_2|)^2 + 4(1 - |z_1 z_2|)|z_1 z_2|^2 < 0.$

In this case, $a \neq b$, $a, b = (((|z_1| - |z_2| \pm \sqrt{D})/(2|z_1 z_2|^2)e^{i \arg z_1}$.

Theorem $(D = (|z_1| - |z_2|)^2 + 4(1 - |z_1 z_2|)|z_1 z_2|^2)$

Let A and B be 3×3 companion matrices with $\sigma = \{z_1, z_2, a\}$ and $\sigma\{z_1, z_2, b\}$, resp., $\ni W(A) = W(B)$. Then

1. $A = B \Leftrightarrow$ *either* $W(A) = W(B) \neq$ *elliptic disc*
   or $W(A) = W(B) =$ *elliptic disc with foci* z_1, z_2
   such that $0 \notin (z_1, z_2)$ *or* $D \geq 0$.
2. $A \neq B \Rightarrow a \neq b$, $a, b = (((|z_1| - |z_2| \pm \sqrt{D})/(2|z_1 z_2|^2)e^{i \arg z_1}$.

Problem (Calbeck 2008)

Let A be an $n \times n$ companion matrix with $\sigma(A) = \{a\}$ such that $W(A)$ is a circular disc, i.e., $\{z : |z - a| \leq \lambda\}$. Then $a =$?

NOTE. If $n = 3$, then $a = 0$ or $|a| = a_0$, a fixed positive number.
More precisely, $a(a^6 + 6a^4 + 3a^2 - 1) = 0$ (Calbeck 2008)

Theorem (Lee & Wang)

If $n > 3$, then $a = 0$, i.e., $A = J_n$.

Proof.

1. $\det(\lambda I_n - Re\ \omega(A - aI_n)) = 0$.
2. Since the coefficients of ω^{n-2} or ω^{n-3} equal zero, we have $a = 0$.
3. $n = 4$: $(-a^2/16)(1 + 4a^2 + 18a^4 + 12a^6 + a^8)\ \omega^2 + \cdots$

Thank you